

ABOUT A BREZIS-MERLE PROBLEM IN DIMENSION 2.

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ABSTRACT. On a bounded open set Ω of \mathbb{R}^2 with smooth boundary, we consider two sequences of functions $(u_i)_i$ and $(V_i)_i$ such that,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega. \end{cases}$$

With, $0 \leq a \leq V_i \leq b < +\infty$.

Assume that,

$$\int_{\Omega} e^{u_i} dx \leq C.$$

If,

$$a > 0,$$

or,

$$a = 0 \text{ and } V_i \rightarrow V \text{ in } C^0(\bar{\Omega}).$$

We have,

$$\|u_i\|_{L^\infty(\bar{\Omega})} \leq c(a, b, C, \Omega).$$

1. INTRODUCTION AND RESULTS.

We set $\Delta = -\partial_{11} - \partial_{22}$ the geometric Laplacian on \mathbb{R}^2 .

On an open set Ω of \mathbb{R}^2 , with a smooth boundary, we consider the following problem:

$$(P) \quad \begin{cases} \Delta u = V e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function V is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V \geq 0$ and $V \in L^p(\Omega)$ or $V e^u \in L^p(\Omega)$ with $p \in [1, +\infty]$.

We can see in [B-M] the two following important Theorems,

Theorem A(Brezis-Merle). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 \leq V_i \leq b < +\infty$ and $\int_{\Omega} e^{u_i} dx \leq C$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, C, K, \Omega)$$

Theorem B(Brezis-Merle). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [S], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C(a/b) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [C-L], an explicit value of $C(a/b) = \sqrt{\frac{a}{b}}$.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [B-L-S]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [L-S] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up.

The questions of Brezis-Merle :

Question 1:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions $(u_i)_i$ solutions of the problem (P) relatively to $(V_i)_i$ with,

$$\begin{cases} 0 < a \leq V_i \leq b < +\infty \\ \int_{\Omega} e^{u_i} dx \leq C'. \end{cases}$$

Is it possible to have,

$$\|u_i\|_{L^\infty(\Omega)} \leq c' = c'(a, b, C', \Omega)?$$

Question 2:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions $(u_i)_i$ solutions of the problem (P) relatively to $(V_i)_i$ with,

$$\begin{cases} 0 \leq V_i \leq b < +\infty \\ V_i \rightarrow V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \leq C'. \end{cases}$$

Is it possible to have,

$$\|u_i\|_{L^\infty(\Omega)} \leq c' = c'(a, b, C', \Omega)?$$

We can find some results which concerning this question. It was proved by Chen-Li, that if we suppose $a = 0$ and $\|\nabla V_i\|_{L^\infty(\Omega)} \leq A$, then, we have a positive answer to the question, see [C-Li].

Also, if we assume $a > 0$ and $\|\nabla V_i\|_{L^\infty(\Omega)} \leq A$, the answer to the question is positive without the assumption on e^{u_i} was given by Ma-Wei, see [M-W].

Here, we have,

Theorem 1. Let us consider three positive numbers a, b, C , and, two sequences of functions, $(u_i)_i$ and $(V_i)_i$, relatively to the problem (P) with the following conditions:

$$\begin{cases} 0 < a \leq V_i \leq b < +\infty \\ \int_{\Omega} e^{u_i} dx \leq C, \quad \forall i \end{cases}$$

Then, there is a positive constant $c = c(a, b, C, \Omega)$ such that,

$$\|u_i\|_{L^\infty(\Omega)} \leq c.$$

Theorem 2. Let us consider two positive numbers a, b , and, two sequences of functions, $(u_i)_i$ and $(V_i)_i$, relatively to the problem (P) with the following condition:

$$\begin{cases} 0 \leq V_i \leq b < +\infty \\ V_i \rightarrow V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \leq C. \end{cases}$$

Then, there is a positive constant $c = c(a, b, C, \Omega)$ such that,

$$\|u_i\|_{L^\infty(\Omega)} \leq c, \quad \forall i.$$

Proof of the Theorem 1 and 2.

Lemma 1:

There is a subsequence $(u_j)_j$ of $(u_i)_i$ and a measurable function u such that:

- 1) $u \geq 0$ a.e in Ω .
- 2) $\int_{\Omega} e^u dx < +\infty$.
- 3) $u_j \rightarrow u$ a.e in Ω .
- 4) $u_j \rightarrow u$ weakly in $W_0^{1,q}(\Omega)$ for all $q \in [1, 2[$.
- 5) $u_j \rightarrow u$ in $L_{loc}^\infty(\Omega)$.

Proof of the Lemma 1:

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We know that,

$$\int_{\Omega} e^{u_i} dx \leq C'.$$

We can use Boccardo-Gallouet Theorem, see [B-G], to have:

$$\|\nabla u_i\|_{L^q(\Omega)} \leq c(q, b, C', \Omega).$$

We use the Rellich-Kondrachov embedding, to extract from (u_i) a subsequence (u_t) which converge in some $L^r(\Omega)$, $r > 1$ to some function u , we know that, we can have a weakly convergence for a subsequence (u_s) of (u_t) . From a strong convergence of (u_s) in $L^r(\Omega)$, $r > 1$, we can extract a subsequence (u_m) which converge to u almost everywhere on Ω , we can see that $u \geq 0$ almost everywhere. We can use the Fatou lemma for e^{u_m} to have $e^u \in L^1(\Omega)$.

Now, we can take a subsequence of exponent $q_n \rightarrow 2$, $q_n < 2$ and, we use the diagonal process, to extract from (u_m) a subsequence (u_n) which converge to u (Ω is bounded), weakly

in $W_0^{1,q}(\Omega)$, $1 \leq q < 2$, almost everywhere and in some L^h for some $h > 1$ large enough. Thus, 1), 2), 3) and 4) are true.

If we use a corollary of Theorem 3 of the Brezis-Merle paper, see [B-M], we can say that (u_m) is in $L_{loc}^\infty(\Omega)$.

First, we consider an exhaustive sequence of compact sets of Ω , (K_l) , we use the diagonal process, to have a subsequence (u_j) of (u_n) which converge in $L^\infty(K_l)$ to a function u_{K_l} for every compact K_l , but (u_j) converge almost everywhere to u and we can conclude that $u_{K_l} = u$ a.e. Thus, (u_j) converge on all compact set K of Ω to u . Thus, 5) is true.

Here, we denote $D(\Omega)$ the set of smooth functions with compact support in Ω . We write $supp(\varphi)$ the support of $\varphi \in D(\Omega)$.

1) Regularity of u .

Case 1: $V_i \rightarrow V$ in $C^0(\bar{\Omega})$

We write:

$$\int_{\Omega} \langle \nabla u_i | \nabla \varphi \rangle dx = \int_{\Omega} V_i e^{u_i} \varphi, \quad \forall \varphi \in D(\Omega).$$

Because $supp(\varphi) \subset \subset \Omega$, and $u_i \rightarrow u$ uniformly on all compact sets of Ω , we have:

$$\int_{\Omega} \langle \nabla u | \nabla \varphi \rangle dx = \int_{\Omega} V e^u \varphi dx, \quad \forall \varphi \in D(\Omega).$$

We know that $u \in W_0^{1,q} \cap L^p(\Omega)$, $\forall p > 1$ and $\int_{\Omega} e^u dx < +\infty$. We can use the corollary 1 of [B-M] to have $\int_{\Omega} e^{ku} dx < +\infty$, and, by the elliptic regularity, we have $u \in C^1(\bar{\Omega})$.

We know that, the regularity of $(u_i)_i$ and $(u_i)_i$ imply that on each compact set, $(\nabla u_i)_i$ converge to a function, but we know that $(u_i)_i$ converge weakly to u in $W_0^{1,q}$, $q > 1$, by the diagonal process and the uniqueness of the weak limit, we can say that $(\nabla u_i)_i$ converge uniformly to ∇u on each compact set of Ω .

Case 2: $0 < a < a < V_i < b$

We know that (V_i) is in $L^\infty(\Omega)$, we can use the *-weak topology and the (*-weakly compactness) Alaoglu-Banach Theorem to have a subsequence (V_i) and $V \in L^\infty(\Omega)$ such that,

$$\int_{\Omega} V_i v dx \rightarrow \int_{\Omega} V v dx, \quad \forall v \in L^1(\Omega).$$

A consequence of the previous result is that $V \geq a$ almost everywhere in Ω . To see this, it is sufficient to consider the functions $v = 1_{\{g \geq a\}}$ and after $v = 1_{\{g \leq a\}}$. Also, we have $g \leq b$

Let $\varphi \in D(\Omega)$, we can use the corollary theorem 3 of [B-M], to have the uniform convergence of $(u_i)_i$ on every compact, also, we use the same arguments as in the previous case 1, the "limit equation" is :

$$\int_{\Omega} \langle \nabla u | \nabla \varphi \rangle dx = \int_{\Omega} V e^u \varphi, \quad \forall \varphi \in D(\Omega).$$

with $0 < a \leq V \leq b$. We obtain the same results as in the previous case 1. The function $u \in C^1(\bar{\Omega})$.

2) The local convergence 1.

Goal

In this part we want to prove that there is a finite number of point $\bar{x}_1, \dots, \bar{x}_m \in \partial\Omega$, such that u_i converge in $C_{loc}^{1,\theta}(\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\})$ to u .

We have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C. \quad (*)$$

Then, according to the Riesz Theorem (see [R]), there is a bounded Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \int_{\partial\Omega} \varphi d\mu, \quad \forall \varphi \in C(\partial\Omega).$$

We call x_0 a non regular point for the measure μ if,

$$\mu(\{x_0\}) \geq 4\pi.$$

According to (*) there is a finite number of non regular points for μ . We denote $\bar{x}_1, \dots, \bar{x}_m$ all the non regular points.

Without loss of , we can suppose the local piece of curve around a point x is an interval of type $[-a, a]$, also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

Let x_0 be a regular point for μ , then, $\mu(\{x_0\}) < 4\pi$. Consider the family of arcs $I_\epsilon =]x_0 - \epsilon, x_0 + \epsilon[\subset \partial\Omega$, we have,

$$1_{I_\epsilon}(x) \xrightarrow{\epsilon \rightarrow 0} 1_{\{x_0\}}(x), \quad 1_{I_\epsilon} \leq 1 \text{ and } \partial\Omega \text{ is compact.}$$

We can use the Lebesgue dominated convergence Theorem for μ to have:

$$\mu(I_\epsilon) \rightarrow \mu(\{x_0\}) \text{ if } \epsilon \rightarrow 0.$$

We construct cutoff function $\eta_\epsilon \in D(\partial\Omega)$ on I_δ , $\delta = \min_{1 \leq i \neq j \leq m} \{d(\bar{x}_i, \bar{x}_j)/2\}$ (we can use charts), as:

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

$$\eta_\epsilon \equiv 1 \text{ on } I_\epsilon \text{ and } \eta_\epsilon \equiv 0 \text{ on } \partial\Omega - I_{2\epsilon}.$$

We extend η_ϵ to a function $\tilde{\eta}_\epsilon$ on $\bar{\Omega}$ as,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0, & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega. \end{cases}$$

Key 1: We can extend η_ϵ to a function $\tilde{\eta}_\epsilon$ explicitly, we take η_ϵ and we translate it with respect to the normal vector in x_0 , after we regularise it like for η_ϵ between ϵ and 2ϵ on $[-a, a] \subset \partial\Omega$.

We know from strong maximum principal and the elliptic regularity, (because $\tilde{\eta}_\epsilon - \tilde{\epsilon} \in C_0^1(\bar{\Omega})$ l'ensemble des fonctions C^1 nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \tilde{\eta}_\epsilon \leq 1, \\ ||\nabla \tilde{\eta}_\epsilon||_{L^\infty(\bar{\Omega})} \leq \frac{C_1}{\epsilon}, \quad C_1 \text{ depends only on } \Omega \text{ and } x_0. \end{cases}$$

Those estimates are easy obtained, because we use the **Key1**, the functions $\tilde{\eta}_\epsilon$ or constant in t , because $\tilde{\eta}_\epsilon(t, x) = \eta_\epsilon(x)$ in the most important part of the new subdomain of $\bar{\Omega}$, also on the corners where we have the estimate of type c/ϵ , $c > 0$.

Now, we write:

$$\Delta[(u_i - u)\eta_\epsilon] = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 < \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon >. \quad (**)$$

Here we want to prove that for $\epsilon > 0$ small enough,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0/2. \quad (1)$$

where $\epsilon_0 > 0$ small enough.

Remark: To obtain our estimate, it is sufficient to choose and to reduce the $I_{2\epsilon}$. In fact we want to find $\tilde{\epsilon} > 0$, and a rank $i' = i'(\epsilon') \in \mathbb{N}$ such we have an uniform estimate for the sequence $(u_i)_i$ on a domain which the boundary in $\partial\Omega$ is $I_{\epsilon'}$.

Here we do not search to tend ϵ to 0 to have an estimate, but only reduce it to have a local uniform estimate.

Step 1: Estimate of the integral of the first term of the right hand side of (**).

We have,

$$\begin{cases} \Delta u = V e^u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (*)'$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between u_i and η_ϵ to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \xrightarrow{i \rightarrow +\infty} \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (**')$$

From (*)' et (**') we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (**''')$$

Remark: In fact, we reduce the interval I_ϵ and we conserve the fact that the integral is strictly smaller than 4π . The fact that $\epsilon > 0$ is small for the interval is not a problem, because our goal is to find an ϵ for which the estimate of the integral in (**) is strictly smaller than 4π .

Step 2: Estimate of integral of the second term of the right hand side of (**).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^2\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

We can construct another hypersurface, more easily. We know that Ω is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the compactness, we can cover the boundary by a finite number of open set of charts, we work locally on open sets of \mathbb{R}_+^2 and we construct step by step a 2-manifold Ω_{ϵ^2} contained in Ω with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of $\Omega - \Omega_{\epsilon^2}$ is $k_2 \leq \mu_L(\Omega - \Omega_{\epsilon^2}) \leq k_1 \epsilon^2$.

Remark: for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^2)$.

We write,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx = \int_{\Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx + \int_{\Omega - \Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx. \quad (***)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$.

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1/\epsilon$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in L^q , $1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$, then, $h = |\nabla u|$ a.e.

If we take $f = 1_{\Omega - \Omega_{\epsilon^2}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \quad i \geq i_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| + \epsilon^2.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq \text{mes}(\Omega - \Omega_{\epsilon^2}) \|\nabla u\|_{L^{\infty}} + \epsilon^2 = \epsilon^2 (k_1 \|\nabla u\|_{L^{\infty}} + 1).$$

Thus,

$$\begin{aligned} \int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx &\leq \|\nabla \tilde{\eta}_{\epsilon}\|_{\infty} \left(\int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| + |\nabla u| \right) \leq \\ &\leq \epsilon C_1 (k_1 \|\nabla u\|_{L^{\infty}} + 1) + C_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 1) \end{aligned}$$

Finally, we have,

$$\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 1) \quad (*)$$

The constant C_1 do not depend on ϵ but on Ω , in its definition.

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$.

We know that, $\Omega_{\epsilon} \subset \subset \Omega$, and $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^2})$

We have,

$$\|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon^2})} \leq \epsilon^2, \text{ for } i \geq i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon^2})} \|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2) \quad (*''')$$

From $(*''')$ and $(*''')$, we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u) \tilde{\eta}_{\epsilon}]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2 + C) \quad (*)$$

We choose $\epsilon > 0$ small enough, we have (1).

We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is $q > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i - u)} dx \leq \int_{\Omega} e^{q(u_i - u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i.$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that:

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write,

$$\Delta(u_i \eta) = V_i e^{u_i} \eta - 2 \langle \nabla u_i | \nabla \eta \rangle + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf\{q_{x_0}, 2\}$. Thus, we can use the elliptic estimates to have $(u_i \eta)_i$ uniformly bounded in $W^{2, q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i \eta)_i$ uniformly bounded in $C^1(\bar{\Omega})$, and if we repeat the previous procedure, we can say that $(u_i \eta)_i$ is uniformly bounded in $C^{1, \theta}(\bar{\Omega})$ for some $\theta \in]0, 1[$.

Finally, we have, for some $\epsilon > 0$ small enough,

$$\|u_i\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

Finally we have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the sequence $(u_i)_i$ is uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

We know that, because $u_i = 0$ on $\partial\Omega$ and $\Delta u_i \geq 0$:

$$\max_{\bar{\Omega}} u_i = u_i(x_i), \quad x_i \in \Omega.$$

Without loss of generality can assume that $x_i \rightarrow \bar{x}_1$.

3) Comparison of the measures.

Here we want to compare the usual measure $d\sigma = d\mu_L$ on $\partial\Omega$ with the new measure μ .

Step 1:

Here, we want to prove that:

$$\int_{\partial\Omega} \partial_\nu u \varphi d\sigma + \sum_{k=1}^m \bar{\mu}_k \varphi(\bar{x}_k) = \int_{\partial\Omega} \varphi d\mu, \quad (2)$$

here, $\bar{\mu}_i = \mu(\{\bar{x}_i\})$.

Proof. It follows from the uniform convergence of our sequence on every compact set (obtained by the diagonal process).

Step 2:

Next, to simplify our computations, we assume that the piece of curve of $\partial\Omega$ are as intervals with the usual Lebesgue measure denoted μ_L . We also write $dx = d\mu_L$ and $\bar{x}_k = k - 1, k = 1, \dots, m$.

Without loss of , we can suppose the local piece of curve around 0 is an interval of type $[-a, a]$, also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

We want to prove that around every blow-up point $k \in \{0, \dots, m\}$,

$$d\sigma = h_k d\mu \text{ with } h \geq 0, \quad h \in L^1([-a_k, a_k], d\mu) \quad (3)$$

where $a_k > 0$ is such that,

$$V_i \geq \alpha_k > 0 \text{ on } [-a_k, a_k] \quad (4)$$

Fundamental remark: (relation between the fact $0 \leq a \leq V_i \leq b$ and $V_i \rightarrow V$ in $C^0(\bar{\Omega})$): To prove locally a realtion between the two measures, we need a condition as in (4), it is the case when we assume $0 \leq a \leq V_i \leq b$. Now we look to the condition, $V_i \rightarrow V$ in $C^0(\bar{\Omega})$.

We know that 0 is a blow-up of $(\partial_\nu u_i)_i$. We take a continuous function φ_ϵ with compact support on $[-a, a]$ such that:

$$0 \leq \varphi_\epsilon \leq 1, \quad \varphi_\epsilon \equiv 1 \text{ on } [-b, b] \text{ with } 2b < a.$$

φ_ϵ vanish outside $[-2b, 2b]$.

$$\begin{cases} \Delta \bar{\varphi}_\epsilon = 0 \text{ dans } \Omega \\ \bar{\varphi}_\epsilon = \varphi_\epsilon. \end{cases}$$

Now, we can use the Green formula between $\bar{\varphi}_\epsilon$ et u_i to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \varphi_\epsilon d\sigma \rightarrow \mu(\varphi_\epsilon) \geq \mu(\{0\}) = \mu_1 > 0$$

The maximum principle imply that $0 = \min_{\bar{\Omega}} \leq \bar{\varphi}_\epsilon \leq \bar{\varphi}_\epsilon \leq \max_{\bar{\Omega}} \bar{\varphi}_\epsilon = 1$.

If $V(0) = 0$, thus,

$$\forall \epsilon' > 0 \exists \delta > 0 \text{ such that } 0 \leq V(x) \leq \epsilon' \text{ on } [-\delta, \delta]$$

But, $V_i \rightarrow V$ in $C^0(\bar{\Omega})$, then,

$$\forall x \in [-\delta', \delta'] \forall i \geq i_0 \quad 0 \leq V_i(x) \leq 2\epsilon'$$

with $\delta' = \inf\{\delta, b\}$.

If we use the fact that $\int_{\Omega} e^{u_i} dx \leq C$, we have,

$$0 < \mu_1 \leq 2C\epsilon' \forall \epsilon' > 0,$$

It is a contradiction if we take $\epsilon' \rightarrow 0$.

Finally,

$$V(0) > 0. \quad (5)$$

The fact that $V_i \rightarrow V$ in $C^0(\bar{\Omega})$, imply that,

$$\exists a', \alpha_0 > 0 \text{ suchthat } V_i(x) \geq \alpha_0 > 0 \text{ on } [-a', a']$$

Now, we want to prove (3).

we try to prove that the measure $d\mu$ is absolutly continous to $d\sigma$:

We want to prove,

$$\mu(A) = 0 \Rightarrow \mu_L(A) = 0.$$

Without loss of , we can suppose the piece of curve is an interval $[-a, a]$, also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

Let $A \subset [-a, a]$ such that $\mu(A) = 0$. The measure μ is regular and we have,

$$0 = \mu(A) = \sup\{\mu(U), U \text{ open set } A \subset U\}.$$

$\forall \epsilon > 0, \exists U_\epsilon$ open set of $[-a, a]$ such that $\mu(U_\epsilon) \leq \epsilon$.

We consider $\omega_\epsilon \subset U_\epsilon$ such that $\mu_L(U_\epsilon - \omega_\epsilon) \leq \epsilon$. (It is possible by an exhaustiv sequence of compact sets of U_ϵ).

We take a continuous function φ_ϵ with compact support on U_ϵ such that:

$$0 \leq \varphi_\epsilon \leq 1, \varphi_\epsilon \equiv 1 \text{ on } \omega_\epsilon.$$

Consider the following system:

$$\begin{cases} \Delta \bar{\varphi}_\epsilon = 0 \text{ dans } \Omega \\ \bar{\varphi}_\epsilon = \varphi_\epsilon. \end{cases}$$

Now, we can use the Green formula between $\bar{\varphi}_\epsilon$ et u_i to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx = \int_{\partial\Omega} \partial_{\nu} u_i \varphi_\epsilon d\sigma \rightarrow \mu(\varphi_\epsilon) \leq \mu(U_\epsilon) \leq \epsilon.$$

Let G be the Green function of the Laplacian on Ω . We can write:

$$\bar{\varphi}_\epsilon(x) = \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma.$$

We can write,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq a \int_{\Omega} V_i e^{u_i} \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma dx.$$

Then,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq a \int_{\{x, d(x, \partial\Omega) \geq \alpha_0\}} \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma dx.$$

With $0 < \alpha_0 \leq \frac{1}{2} \sup\{d(x, y), x, y \in \bar{\Omega}\}$.

We use the definition of the Green function,

$$\Delta G(x, y) = \delta_x, \quad G(x, y) = 0 \text{ on } \partial\Omega.$$

Let,

$$E_1 = \{x, d(x, \partial\Omega) \geq \alpha_0 > 0\} \text{ and } E_2 = \{y, d(y, \partial\Omega) \leq \alpha_0/2\}.$$

We have $E_1 \cap E_2 = \{\emptyset\}$ and by the strong maximum principle, we have,

$$\partial_{\nu, y} G(x, y) \geq \beta_0 > 0 \quad \forall x \in E_1, y \in E_2.$$

Then,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq m_0 \mu_L(\omega_\epsilon).$$

We can write,

$$m_0 \mu_L(\omega_\epsilon) = m_0 [\mu_L(U) - \epsilon] \leq \mu(U_\epsilon) + \epsilon \leq 2\epsilon.$$

Then,

$$\mu_L(A) \leq \mu_L(U_\epsilon) \leq \frac{(2 + m_0)\epsilon}{m_0}.$$

Finaly,

$$\mu_L(A) = 0.$$

If we use the Radon-Nikodym theorem we obtain (3).

4) The Local convergence 2

Next, to simplify our computations, we assume that the piece of curve of $\partial\Omega$ are as intervals with the usual Lebesgue measure denoted μ_L . We also write $dx = d\mu_L$ and $\bar{x}_k = k - 1, k = 1, \dots, m$.

Without loss of , we can suppose the local piece of curve around $x_0 = 0$ is an interval of type $[-a, a]$, also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

We follow the method of the "local convergence 1", we choose other type of function η_ϵ .

$$\begin{cases} \eta_\epsilon(x) = \left(\frac{|x|}{\epsilon}\right)^{2/3}, & \text{on } |x| \leq \epsilon, \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 1, & \text{on } [\epsilon, 2\epsilon] \\ \eta_\epsilon = -\frac{x}{\epsilon} + 3, & \text{on } [\epsilon, 2\epsilon] \\ \eta_\epsilon \equiv 0, & \text{outside } [3\epsilon, a], \\ 0 \leq \eta_\epsilon \leq 1, \end{cases}$$

A computation gave:

$$\|\eta'_\epsilon\|_{W^{1,5/2}([-3\epsilon, 3\epsilon])} \leq C_1 \epsilon^{-\alpha}, \quad \alpha > 0 \quad (6)$$

Remark about the exponent 5/2, here our functions η_ϵ are not C^1 , but in $W^{1,5/2}$ with $5/2 > 2$, and when we use the Hölder inequality in

$\int_\Omega < \nabla(u_i - u) | \tilde{\eta}_\epsilon$ we can use many argument to have this quantity small.

We extend η_ϵ to a function $\tilde{\eta}_\epsilon$ on $\bar{\Omega}$ as,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0, & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega. \end{cases}$$

As in "The local convrgence 1" we have the:

Key 2: We can extend η_ϵ to a function $\bar{\eta}_\epsilon$ explicitly, we take η_ϵ and we translate it with respect to the normal vector in x_o , after we regulise it like for η_ϵ between ϵ and 2ϵ on $[-a, a] \subset \partial\Omega$.

We know from strong maximum principal and the elliptic regularity, (because $\bar{\eta}_\epsilon - \tilde{\eta}_\epsilon \in C_0^1(\bar{\Omega})$) l'ensemble des fonctions C^1 nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \tilde{\eta}_\epsilon \leq 1, \\ \|\nabla \tilde{\eta}_\epsilon\|_{L^{1,5/2}(\Omega)} \leq \frac{C_1}{\epsilon^\beta}, \beta > 0 \end{cases} \quad C_1 \text{ depends only on } \Omega \text{ and } x_0 \quad (7).$$

Those estimates are easy obtained, because we use the **Key2**, the functions $\bar{\eta}_\epsilon$ or constant in t , because $\bar{\eta}_\epsilon(t, x) = \eta_\epsilon(x)$ in the most important part of the new subdomain of $\bar{\Omega}$, also on the corners where we have the estimate of type $c/\epsilon^s, s > 0, c > 0$.

Now, we write:

$$\Delta[(u_i - u)\eta_\epsilon] = (V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon - 2 < \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon >. \quad (8)$$

Here we want to prove that for $\epsilon > 0$ small enough,

$$\int_\Omega |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0/2. \quad (9)$$

where $\epsilon_0 > 0$ small enough.

Remark: To obtain our estimate, it is sufficient to choose and to reduce the $I_{2\epsilon}$. In fact we want to find $\tilde{\epsilon} > 0$, and a rank $i' = i'(\epsilon') \in \mathbb{N}$ such we have an uniform estimate for the sequence $(u_i)_i$ on a domain which the boundary in $\partial\Omega$ is $I_{\epsilon'}$.

Here we do not search to tend ϵ to 0 to have an estimate, but only reduce it to have a local uniform estimate.

We can use the (1) and (2) to have,

$$\int_{-3\epsilon}^{3\epsilon} (\partial_\nu u_i) \eta_\epsilon(x) dx \rightarrow \int_{-3\epsilon}^{3\epsilon} \eta_\epsilon(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_\nu u) \eta_\epsilon(x) \leq 6\epsilon \|\partial_\nu u\|_{L^\infty}.$$

Step 1: Estimate of the integral of the first term of the right hand side of (9).

We have,

$$\begin{cases} \Delta u = V e^u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (*)$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between u_i and η_ϵ to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \xrightarrow{i \rightarrow +\infty} \mu(\eta_\epsilon) = \int_{-3\epsilon}^{3\epsilon} \eta_\epsilon(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_\nu u) \eta_\epsilon(x) \leq 6\epsilon \|\partial_\nu u\|_{L^\infty}. (**)$$

From (*) et (**) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (***)$$

Remark: In fact, we reduce the interval I_ϵ and we conserve the fact that the integral is strictly smaller than 4π . The fact that $\epsilon > 0$ is small for the interval is not a problem, because our goal is to find an ϵ for which the estimate of the integral in (**) is strictly smaller than 4π .

Step 2: Estimate of the integral of the second term of the right hand side of (8).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^2\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

We can construct another hypersurface, more easily. We know that Ω is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the compactness, we can cover the boundary by a finite number of open set of charts, we work locally on open sets of \mathbb{R}_+^2 and we construct step by step a 2-manifold Ω_ϵ contained in Ω with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of $\Omega - \Omega_{\epsilon^2}$ is $\mu_L(\Omega - \Omega_\epsilon) \leq k_1 \epsilon^\gamma$.

Key 3: The choice of γ is linked to β , we choose it later where appear $\mu_L(\Omega - \Omega_\epsilon)$ and $\epsilon^{-\beta}$.

We write,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx = \int_{\Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx + \int_{\Omega - \Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx. \quad (***)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$.

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{5/2}} \leq C_1/\epsilon^{\beta}$, C_1 depends on Ω

We know that $(|\nabla u_i|^{5/3})_i$ is bounded in $L^{3/2}$, $1 < 3/2 < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$, then, $h = |\nabla u|^{5/3}$ a.e.

If we take $f = 1_{\Omega - \Omega_{\epsilon_2}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1 \int_{\Omega - \Omega_{\epsilon}} |\nabla u_i|^{5/3} \leq \int_{\Omega - \Omega_{\epsilon}} |\nabla u|^{5/3} + \epsilon^{\gamma} \leq (1 + \|\nabla u\|_{L^{\infty}}^{5/3}) \epsilon^{\gamma}.$$

If we use the Hölder inequality we have,

$$\int_{\Omega - \Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \|\nabla \tilde{\eta}_{\epsilon}\|_{L^{5/2}} \times \left[\int_{\Omega - \Omega_{\epsilon}} (|\nabla u_i| + |\nabla u|)^{5/3} dx \right]^{3/5} \leq$$

The Minkowski give,

$$\int_{\Omega - \Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq C_2 \epsilon^{-\beta} \left[\left(\int_{\Omega - \Omega_{\epsilon}} (|\nabla u_i|^{5/3})^{3/5} \right)^{3/5} + \left(\int_{\Omega - \Omega_{\epsilon}} |\nabla u|^{5/3} dx \right)^{3/5} \right]$$

Thus,

$$\int_{\Omega - \Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq C_3 \epsilon^{5/3\gamma - \beta}$$

Il suffit de prendre $5/3\gamma - \beta > 0$ pour avoir une estimation negligeeable de l'intégrale $\int_{\Omega - \Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$

Step 2.2: Estimate of $\int_{\Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$.

We know that, $\Omega_{\epsilon} \subset \subset \Omega$, and $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon_2})$

We have,

$$\|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon_2})} \leq \epsilon_{\beta+1}, \text{ for } i \geq i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon_2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon_2})} \|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2) \quad (****)$$

From $(****)$ and $(****)$, we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\begin{aligned} \int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx &\leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2 + C) \quad (\hat{*}) \\ &\leq \epsilon C_1 (k_1 \|\nabla u\|_{L^{\infty}} + 1) + C_1 \int_{\Omega - \Omega_{\epsilon_2}} |\nabla u| \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 1) \end{aligned}$$

Finally, we have,

$$\int_{\Omega - \Omega_{\epsilon^2}} |< \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon >| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (*)$$

The constant C_1 do not depend on ϵ but on Ω , in its definition.

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2\}$,

$$\int_{\Omega} |< \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon >| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (*''')$$

From $(*''')$ and $(*''')$, we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (\hat{*})$$

We choose $\epsilon > 0$ small enough, we have (1).

We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is $q > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i - u)\eta_\epsilon} dx \leq \int_{\Omega} e^{q(u_i - u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0) = B(x_0, \epsilon) \cap \Omega$ is a neighborhood of x_0 in $\bar{\Omega}$.

We have,

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}(u_i - u)\eta_\epsilon} dx \leq C, \quad \forall i.$$

Now, we consider a function $\eta \in C^\infty(\mathbb{R}^2)$ such that:

$$\Delta(u_i \eta) = V_i e^{u_i} \eta - 2 < \nabla u_i | \nabla \eta > + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf q_{x_0}, 2$. Thus, we can use the elliptic estimates to have $(u_i \eta_\epsilon)_i$ uniformly bounded in $W^{2,q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i \eta_\epsilon)_i$ uniformly bounded in $C^1(\bar{\Omega})$, and if we repeat the previous procedure, we can say that $(u_i \eta_\epsilon)_i$ is uniformly bounded in $C^{1,\theta}(\bar{\Omega})$ for some $\theta \in]0, 1[$.

Finally, we have, for some $\epsilon > 0$ small enough,

$$\|u_i \eta_\epsilon\|_{C^{1,\theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We know that, because $u_i = 0$ on $\partial\Omega$ and $\Delta u_i \geq 0$:

$$\max_{\Omega} u_i = u_i(x_i), \quad x_i \in \Omega.$$

Without loss of generality can assume that $x_i \rightarrow \bar{x}_1$.

We take the convention $x_0 = 0$. We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is $q > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i-u)\tilde{\eta}_\epsilon} dx \leq \int_{\Omega} e^{q(u_i-u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$.

There is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i \eta_\epsilon} dx \leq C, \quad \forall i.$$

Now, we consider a cutoff function $\eta \in C_0^\infty(\mathbb{R}^2)$ (smooth functions with compact support) such that:

$$\Delta(u_i \eta) = V_i e^{u_i} \eta - 2 \langle \nabla u_i | \nabla \eta \rangle + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf\{q_{x_0}, 2\}$. Thus, we can use the elliptic estimates to have $(u_i \eta)_i$ uniformly bounded in $W^{2, q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i \eta)_i$ uniformly bounded in $C^1(\bar{\Omega})$, and if we repeat the previous procedure, we can say that $(u_i \eta)_i$ is uniformly bounded in $C^{1, \theta}(\bar{\Omega})$ for some $\theta \in]0, 1[$.

Finally, we have, for some $\epsilon > 0$ small enough,

$$\|u_i \tilde{\eta}_\epsilon\|_{C^{1, \theta}[\bar{B}(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

On $I_\epsilon = [-\epsilon, \epsilon]$, because $u_i = 0$ on I_ϵ ,

$$\text{For } y \in \Omega \quad |\nabla(u_i \eta_\epsilon)(y)| = |\nabla u_i(x) \eta_\epsilon(y) + u_i(y) \nabla \eta_\epsilon(y)| \leq c_3$$

If we tend $y \rightarrow x \in I_\epsilon - \{0\}$, ($\tilde{\eta}_\epsilon(x) \rightarrow \eta_\epsilon(x)$, $x \neq 0$ and $u_i(x) = 0$) we obtain,

$$|\nabla u_i(x)| = \partial_\nu u_i(x) \leq c_4 |x|^{-2/3}, \quad \text{on } [-\epsilon, \epsilon] - \{0\}$$

Thus,

$$\int_{-\epsilon}^{\epsilon} [\partial_\nu u_i(x)]^{4/3} dx \leq c_5 \quad \forall i \geq i_4, \quad i_4 \in \mathbb{N}$$

But,

$$\partial_\nu u_i \rightarrow \partial_\nu u, \quad \mu_L \text{ a.e.}$$

After a subsequence, $\partial_\nu u_i$ converge weakly to $\partial_\nu u$, thus,

$$\int_{-\epsilon}^{\epsilon} \partial_\nu u_i \varphi dx \rightarrow \int_{-\epsilon}^{\epsilon} \partial_\nu u \varphi, \quad \forall \varphi \in C_c([-\epsilon, \epsilon])$$

Where, $C_c([-\epsilon, \epsilon])$ is a set of continuous function with compact support.

But, we know,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx = \int_{-\epsilon}^{\epsilon} \partial_\nu u_i \varphi dx \rightarrow \varphi d\mu.$$

Also, we know that,

$$dx = h d\mu,$$

Then,

$$\int_{-\epsilon}^{\epsilon} [(\partial_\nu u)h - 1] d\mu \varphi = 0 dx, \quad \forall \varphi \in C_c([-\epsilon, \epsilon])$$

We conclude that,

$$(\partial_\nu u)h = 1. \quad \mu \text{ a.e.}$$

It means that,

$$\mu_1 = 0 \quad \text{and} \quad h(0).$$

It is a contradiction.

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References.

- [B-G] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. 87 (1), (1989), 149-169.
- [B-L-S] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [B-M] H. Brezis, F. Merle. Uniform estimates and Blow-up Behavior for Solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).
- [C-Li] W. Chen, C. Li. A priori Estimates for solutions to Nonlinear Elliptic Equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
- [C-L] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [J] J. Jost. Partial Differential Equations. Springer-Verlag 1998.
- [L] YY. Li, Harnack type Inequality, the Methode of Moving Planes. Commun. Math. Phys. 200 421-444.(1999).
- [L-S] YY. Li, I. Shafrir. Blow-up Analysis for Solutions of $-\Delta u = Ve^u$ in Dimension Two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.
- [M-W] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.
- [R] W. Rudin. Real and Complex Analysis.
- [S] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

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